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10.2 $A := \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$A - \lambda \cdot I = 0$$

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$\det\left(\begin{pmatrix} 4-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}\right) = (4-\lambda)\cdot(4-\lambda)\cdot(2-\lambda) + 0 + 0 - 0 - 0 - (2-\lambda)\cdot 1\cdot 1$$

$$= -\lambda^3 + 10\lambda^2 - 31\lambda + 30$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 5 \leftarrow \text{Eigenwerte}$$

1\. Eigenvektor:
$$\left.\begin{aligned} (4-2)x_1 + x_2 &= 0 \\ x_1 + (4-2)x_2 &= 0 \\ (2-2)x_3 &= 0 \end{aligned}\right\} \Rightarrow EV_1 := \left\{ \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$$

2\. Eigenvektor:
$$\left.\begin{aligned} (4-3)x_1 + x_2 &= 0 \\ x_1 + (4-3)x_2 &= 0 \\ (2-3)x_3 &= 0 \end{aligned}\right\} \Rightarrow EV_2 := \left\{ \mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R} \right\}$$

3\. Eigenvektor:
$$\left.\begin{aligned} (4-5)x_1 + x_2 &= 0 \\ x_1 + (4-5)x_2 &= 0 \\ (2-5)x_3 &= 0 \end{aligned}\right\} \Rightarrow EV_3 := \left\{ \upsilon \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \upsilon \in \mathbb{R} \right\}$$

$$\Rightarrow B = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

3/3

10.3) a) $B := \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

B ist ONB, somit gilt: $B^{-1} = B^T$

$$B^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$B \cdot \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \cdot B^{-1} = B \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot B^{-1}$$

$$= \begin{pmatrix} 5/3 & 1/3 & -2/3 \\ 1/3 & 8/3 & -1/3 \\ -2/3 & -1/3 & 5/3 \end{pmatrix} = A \quad ✓$$

b) $$A \cdot \vec{b}_1 = \begin{pmatrix} \frac{5}{3\cdot\sqrt{2}} - \frac{2}{3\cdot\sqrt{3}} \\ \frac{1}{3\cdot\sqrt{3}} - 3\cdot\frac{1}{\sqrt{2}} \\ \frac{-2}{3\cdot\sqrt{2}} + \frac{5}{3\cdot\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad ✓$$

3/3

10.4 a) $\lim_{n\to\infty}\left(\frac{4n^6+2n^3+5}{7n^3+2n^6+13}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n^6}\left(4+\frac{2n^3}{n^6}+\frac{5}{n^6}\right)}{\cancel{n^6}\left(2+\frac{7n^3}{n^6}+\frac{13}{n^6}\right)}\right)$

$\frac{2n^3}{n^6} \to 0$, $\frac{5}{n^6} \to 0$, $\frac{7n^3}{n^6} \to 0$, $\frac{13}{n^6} \to 0$

$= \lim_{n\to\infty}\left(\frac{4}{2}\right) = 2$ ✓

b) $\lim_{n\to\infty}\left(\frac{42n^2+23n+12}{31n^3+17n+\pi}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n^2}\left(42+\frac{23n}{n^2}+\frac{12}{n^2}\right)}{\cancel{n^2}\left(31n+\frac{17n}{n^2}+\frac{\pi}{n^2}\right)}\right)$

$\frac{23n}{n^2} \to 0$, $\frac{12}{n^2} \to 0$, $\frac{17n}{n^2} \to 0$, $\frac{\pi}{n^2} \to 0$

$= \lim_{n\to\infty}\left(\frac{42}{31n}\right) = 0$ ✓

c) $\lim_{n\to\infty}\left(\frac{e^{-n}}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{e^n n^2}\right) = 0$ ✓

$e^n n^2 \to \infty$

d) $\lim_{n\to\infty}\left(\frac{e^n}{n^2}\right) = +\infty$, da eine exponentielle Fnk. schneller wächst als eine quadratische Fnk. ✓

4/4

10.5 a) $\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ ✓

b) $\sum_{k=3}^{\infty} \frac{1}{4^k} = \left(\sum_{k=0}^{\infty} \frac{1}{4^k}\right) - \left(\sum_{k=0}^{2} \frac{1}{4^k}\right)$

$= \frac{1}{1-\frac{1}{4}} - \left(\frac{1}{4^0}+\frac{1}{4^1}+\frac{1}{4^2}\right)$

3/3

$= \frac{1}{48}$ ✓

10.6) a) $\sum_{k=1}^{\infty} \cancel{\frac{1}{k^2 \cdot 2^k}}\ \frac{2^k}{k^2}$, da 2^k schneller wächst als k^2 => divergiert ✓

b) Konvergiert gegen 0. ✓

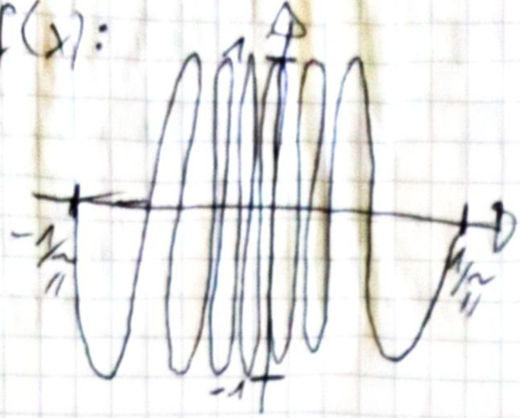
c) Divigiert. Siehe Harmonische Reihe. ✓

3/3

$\lim_{n\to\infty} e^{-n} \cdot \frac{1}{n^2}$

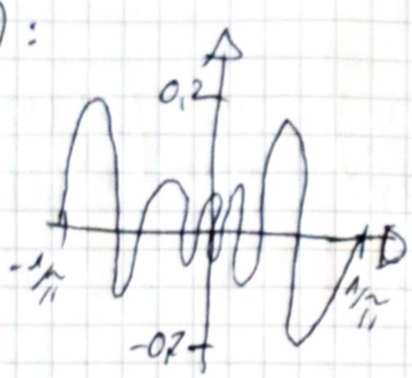
$e^{-n} \to 0$, $\frac{1}{n^2} \to 0$

10.7) a) $f(x)$:

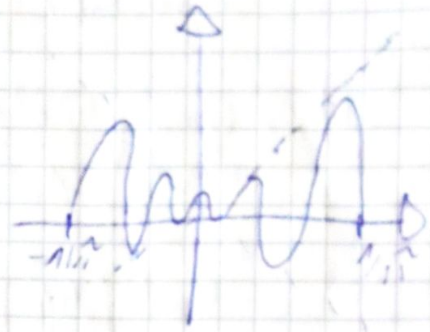


✓

$g(x)$:



(✓)



b) Ja, denn $f(x)$ und $g(x)$ streben gegen Null $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0$ ↯ . Da $\sin\left(\frac{1}{x}\right)$ in beiden Fnk. vorkommt, streben beide gegen 0. Die Null selbst ist nicht enthalten, da man nicht durch 0 teilen darf. Diese wird jedoch durch einen Sonderfall gegeben.

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$\Sigma: 18/19$

b) z.z. $\lim_{x \to 0} f(x) = f(0) \Rightarrow \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0$

$\lim_{a \to \infty} \sin(a) = 0$ ↯

g: z.z.: $\lim_{x \to 0} g(x) = g(0) \Rightarrow \lim_{x \to 0} \underbrace{|x|}_{0} \cdot \underbrace{\sin\left(\frac{1}{x}\right)}_{[-1,1]} = 0$ ✓